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DEFINITE 4-MANIFOLDS

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1. Introduction

The paucity of positive definite unimodular integral bilinear forms which are realized as the intersection form of a closed smooth 4-manifold is demonstrated by the following recent theorem of S. Donaldson:

Theorem (*Donaldson* [4]). *Let X be a smooth closed oriented 4-manifold with positive definite intersection form θ. Then θ is "standard"; i.e. over the integers $\theta \cong (1) \oplus \cdots \oplus (1)$.*

This theorem was originally proved under the assumption that X is simply connected [2], and has also been extended by M. Furuta [7] to cover X with $H_1(X; \mathbf{Z}) = 0$ by techniques similar to those used in [4]. The proofs of all these versions of the theorem rely on quite detailed ad hoc analysis and on the deep and difficult work of C. Taubes [9] (cf. [8]).

We have long felt that it would be worthwhile to give a proof of Donaldson's theorem which reduced the role played by analysis and thus be more accesible to topologists. Our work in [5] was a start in that direction. The purpose of this paper is to give a proof of Donaldson's theorem under the assumption that $H_1(X; \mathbf{Z})$ has no 2-torsion while using as analytical input only the basic work of K. Uhlenbeck [10], [11]. Our proof is in spirit similar to that of [5], using SO(3)-connections, but makes more apparent the importance of the "basepoint fibration" (see §2). By combining our techniques with Donaldson's study of orientations of moduli spaces one can presumably remove our hypothesis on $H_1(X; \mathbf{Z})$ as in [4, 4(c)]; however we do not know an elementary argument that will remove this hypothesis.

As in [5] we base our proof on a useful characterization of nonstandard integral inner product spaces. Let W be a positive definite unimodular integral inner product space and define an equivalence relation on W by declaring that $w_1 \sim w_2$ if $w_1 \equiv w_2 \pmod 2$ and $w_1^2 = w_2^2$. Note that $-w \sim w$. Set $\mu(w) = \frac{1}{2}\#(w' \in W | w' \sim w)$ and call an element $e \in W$ *minimal* if $e^2 \leq w^2$ for all $w \equiv e \pmod 2$.

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