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TOTAL ABSOLUTE CURVATURE AND EMBEDDED MORSE NUMBERS

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Abstract

In this paper we use techniques of Morse theory to compute, under mild hypotheses, the infimum of the total absolute curvatures $\inf \tau(M^m \subset \mathbf{R}^w)$ for the smooth embeddings $M^m \subset \mathbf{R}^w$ in a given isotopy class.

1. Introduction

In 1929, W. Fenchel [8] showed that a circle immersed in $\mathbf{R}^3$ has (normalized) total absolute curvature (cf. §2 for the definition) at least 2 with equality only for the boundary of a convex planar disc. This was followed in 1949 by work of Fary [7] and Milnor [19] who showed that a knot in $\mathbf{R}^3$ has total absolute curvature more than 4. Since that time there has been considerable effort to obtain lower bounds for the total absolute curvature τ of a closed manifold immersed or embedded in Euclidean space in terms of the topological invariants of the situation and to study the consequences of small curvature (cf. e.g. Borsuk [1], Chern & Lashof [3], [4], Ferus [9], Fox [10], Kuiper & Meeks [15], Langevin & Rosenberg [17], Meeks [18], Pinkall [28], Sunday [32], and Wintgen [37]).

Recall that a Morse function on a smooth compact manifold M is a smooth real valued function on M whose critical points are all nondegenerate. The Morse number $\mu(M)$ is the minimum of the number of critical points of the Morse functions on M. For $m \neq 3$, 4 or 5 this is the same as the number of cells in the smallest CW complex with the simple homotopy type of M (cf. Appendix 2.7). In 1958 Chern and Lashof [4] proved that for an immersion $\tau(i) \geq \mu(M)$, and raised the problem of determining the infimum of $\tau(i)$ as i varies over some class of maps, such as all immersions, a regular homotopy class of immersions, all embeddings, or an isotopy class of embeddings. In particular they formulated:

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