J. DIFFERENTIAL GEOMETRY
27 (1988) 115–136

AN INDEX THEOREM ON OPEN MANIFOLDS. II

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Introduction

This paper is the sequel to [19], in which we proved an abstract index theorem for Dirac-type operators on certain noncompact manifolds. Here we will give some concrete applications of this result, and will also discuss its relationship with L^2 index theorems of Atiyah and Connes.

The set-up for [19] is as follows. Let M be a noncompact oriented Riemannian manifold of bounded geometry, and D a Dirac operator of bounded geometry over it. D is equipped with a grading η, and it will be convenient to use the notations D^+ and D^- for the restrictions of D to the $+1$ and -1 eigenspaces of η. Suppose that M admits a regular exhaustion with corresponding fundamental class $\mathcal{m}$ and trace functional τ. Then the main theorem of [19] computes

$$\dim_\tau(\operatorname{Ind} D) = \langle \mathbf{I}(D), \mathcal{m} \rangle;$$

it identifies a "real-valued index" of D with a "topological" invariant. The fundamental question studied here is: How does the number $\dim_\tau(\operatorname{Ind} D)$ relate to the kernel of D?

Recall from [19, 8.1] the equation

$$\dim_\tau(\operatorname{Ind} D) = \tau(\phi(D^-D^+)) - \tau(\phi(D^+D^-))$$

where ϕ is any Schwartz-class function on $\mathbf{R}^+$ with $\phi(0) = 1$. If the manifold M were compact, one could argue as follows: D has discrete spectrum, hence there is a smooth ϕ of compact support such that $\phi(0) = 1$ and $\phi(\lambda) = 0$ for all nonzero eigenvalues λ of D^2. Then $\phi(D^-D^+)$ is the projection P^+ onto the kernel of D^+, and similarly $\phi(D^+D^-)$ is the corresponding projection P^-, and one gets

$$\dim_\tau(\operatorname{Ind} D) = \tau(P^+) - \tau(P^-), \tag{0.1}$$

Received November 4, 1985 and in revised form, November 25, 1986.