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AREA AND THE LENGTH OF THE SHORTEST CLOSED GEODESIC

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1. Introduction

The main purpose of this paper is to prove the following theorem:

Theorem 4.2. *For any metric on a two-dimensional sphere* $31\sqrt{A} \geqslant L$, *where* A *represents the area, and* L *the length of the shortest nontrivial closed geodesic.*

The constant 31 above is not the best constant. One suspects that the best constant would be $3^{1/4}2^{1/2} \simeq 1.86121$. We will discuss this later.

The question, for which the above theorem is the answer in two dimensions, was posed by Gromov for all closed manifolds in [12, p. 135]. The corresponding theorem is known for all other closed surfaces as we will see below. The difficulty with the sphere is that it is simply connected. In fact, all other results relating area (or volume) to the length of closed geodesics concern essential (not null homotopic) geodesics. That is, they concern $\operatorname{sys}(M)$ (read "systole of M"), the length of the shortest essential geodesic.

The first theorem of this type was proved by Loewner in 1949 in an unpublished work (see [3] and [4]). He showed that for any metric on the two torus $3^{1/4}2^{1/2}\sqrt{A(M)} \geqslant \operatorname{sys}(M)$ with equality holding if and only if M is a flat equilateral torus. (The fact that the constant is the same as the conjectured constant for S^2 comes from the fact that both extremal metrics are built out of two flat equilateral triangles.) The proof of the theorem relies on the fact that all metrics are conformal to a flat metric. Using a similar method Pu in 1952 (see [17]) showed that for any metric on $\mathbf{RP}^2$; $\sqrt{\pi/2}\sqrt{A(M)} \geqslant \operatorname{sys}(M)$, with equality holding if and only if M has constant curvature.

In 1960 Accola [1] and Blatter [6] independently showed that there was a function $f(g)$ such that for any metric on a surface of genus g, $f(g)\sqrt{A(M)} \geqslant \operatorname{sys}(M)$. Unfortunately, as g tends to ∞ the function $f(g)$ tends to ∞ while one would expect it to tend to 0. In 1981 Hebda [15] and independently

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