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η-INVARIANTS, THE ADIABATIC APPROXIMATION AND CONICAL SINGULARITIES

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PART I: THE ADIABATIC APPROXIMATION

0. Introduction

In this paper, we discuss a remarkable formula derived by Witten for the η-invariant of a mapping torus, $Y^{4k-2} \to N^{4k-1} \to S^1$ (see (1.56), Theorem 4.27 and [26, §IV]). Witten's derivation is based on the adiabatic approximation as it is often applied in quantum mechanics and is not rigorous. Here, in Part I, we treat explicitly the case of signature operators by heat equation methods, using Duhamel's principle and the techniques of [13] (see also [14]).[1] Bismut and Freed independently treat the case of Dirac operators, using the heat equation and probability theory (see [4], [5]).

Witten's formula is closely related to the work of Atiyah-Donnelly-Singer [1] on η-invariants of solvmanifolds (as others have independently observed). In Appendix 3 to Part I, we show how to obtain a quick proof of a similar result for the case of a 1-dimensional base space, by starting with Witten's formula. We will discuss the generalization of Witten's formula to higher dimensional base spaces and its application to η-invariants of higher dimensional solvmanifolds elsewhere.

In Part II, we discuss in detail the relation between the result of Part I and our previous work on analysis on spaces with conical singularities. The expression in Witten's formula (which is used to define the notion of *anomaly* in physics) arose there when we considered the variational derivative of the η-invariant for a family of spaces (X^{4k-1}, g_u) with conical singularities. The discussion of Part II shows that this expression is also equal to the contribution at a singular stratum, Σ^1, of dimension 1, when one calculates the L_2-signature

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[1] The idea of the proof is described in more detail in §1.