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# GEOMETRY OF MAPS BETWEEN GENERALIZED FLAG MANIFOLDS

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A classical problem in Differential Geometry is the construction and classification of *minimal immersions* between Riemannian manifolds. A recently studied variant of this problem is obtained on replacing minimal immersions by *harmonic maps*. If the manifolds are homogeneous with respect to the actions of certain Lie groups, and are equipped with homogeneous Riemannian metrics, one naturally expects the theory of Lie groups to play a role. We shall use this principle, in a very specific situation, to produce new examples of harmonic maps.

In §1 we summarize well-known facts concerning homogeneous geometric structures on homogeneous spaces $G/H$, and in §2 we discuss the second fundamental form of a map $G/H \to G'/H'$ which is induced by a homomorphism $\theta: G \to G'$. Our basic result appears in §3 (Theorem 3.4), this being a necessary and sufficient algebraic condition for such a map to be harmonic, when $G/H$ and $G'/H'$ are generalized flag manifolds and $G'$ is the unitary group. An important example is discussed in §4, namely that of the "higher order Gauss maps" of the maximal projective weight orbit of an irreducible representation of $G$. The harmonic maps which arise this way are very special, being "homogeneous," but in §5 we show how they may be modified to produce large families of "nonhomogeneous" examples.

Lest this program seem too uninspiring, we shall attempt to give some justification and to point out connections with other problems of current interest. The principal motivation was provided by the paper [15], in particular the results concerning harmonic maps from $\mathbf{C}P^1$ to $\mathbf{C}P^n$, and the author is grateful to Professor James Eells and John Wood for discussing their results. If $f: \mathbf{C}P^1 \to \mathbf{C}P^n$ is holomorphic, there are the well-known holomorphic "associated curves" $f_0, f_1, f_2, \cdots$ (see [18, Chapter 2, §4]); $f_i$ is the map from $\mathbf{C}P^1$

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