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COMPLETE LEFT-INVARIANT AFFINE STRUCTURES ON NILPOTENT LIE GROUPS

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0. Introduction

A smooth manifold M^n is called an affine manifold if it admits a torsion free affine connection whose curvature tensor is zero, or if it admits a coordinate system whose coordinate transition homeomorphisms are affine transformations in $\mathbf{R}^n$. For example, Riemannian flat or Lorentz flat manifolds are subclasses of such manifolds.

The manifold M is said to be complete if every geodesic can be defined on all time intervals. By a well-known theorem the connected complete affine manifolds are just the quotients $\mathbf{R}^n/\Gamma$, where Γ is a subgroup of $\mathrm{Aff}(\mathbf{R}^n)$, the group of all the affine transformations of $\mathbf{R}^n$, acting freely and properly discontinuously on $\mathbf{R}^n$.

The group theoretic nature of such Γ is an open question and it is suggested that such Γ should be virtually polycyclic (i.e., contains a subgroup of finite index which is polycyclic) (see [1], [2], [16].) Milnor showed the converse, namely every virtually polycyclic group can be realized by such a Γ [16].

Recently Fried and Goldman, following an idea of Auslander, showed that such Γ, assuming it is virtually polycyclic, is virtually contained in a connected Lie subgroup G of $\mathrm{Aff}(\mathbf{R}^n)$ which acts simply transitively on $\mathbf{R}^n$ (so G is homeomorphic to $\mathbf{R}^n$) [8]. It is well known that a simply connected Lie group G acts simply transitively on $\mathbf{R}^n$ as affine transformations iff G admits a complete left-invariant affine structure.

Conversely, if G has a complete left-invariant affine structure and Γ is a discrete subgroup, then $\Gamma \setminus G$ will become an affine manifold.

Therefore, to identify affine manifolds, it is natural to ask which simply connected Lie groups G admit a complete left-invariant affine structure. It is known that those G which act simply transitively on $\mathbf{R}^n$ as affine transformations are solvable [2], [16].

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