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A REGULARITY THEOREM FOR HARMONIC MAPS WITH SMALL ENERGY

GUOJUN LIAO

1. Introduction

This paper studies the regularity problem of harmonic maps in higher dimensions. We consider maps from the unit ball B in $\mathbf{R}^n$ ($n > 2$) equipped with a metric g into a compact submanifold N^m of $\mathbf{R}^k$. We say that $u \in L_1^2(B, N)$ if $u \in L_1^2(B, \mathbf{R}^k)$ and $u(x) \in N$ a.e $x \in B$. The energy $E(u)$ of u is defined as $E(u) = \int_B |\nabla u|^2 \, dv$. A weakly harmonic map is defined to be the weak solution to the formal Euler-Lagrange equations, which form a nonlinear elliptic system. The equations are

$$(1.1) \qquad \Delta u^i(x) = g^{\alpha\beta}(x) A^i\left(\frac{\partial y}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right), \qquad i = 1, 2, \cdots, K,$$

where $A_u(X, Y) \in (T_u N)^\perp$ is the second fundamental form of N given by $A_u(X, Y) = (D_X Y)^\perp$. X, Y are vector fields on N in a neighborhood of $u \in N$.

It is easy to see that u is harmonic if and only if $(d/dt)E(u_t)|_{t=0} = 0$, where u_t is a 1-parameter family of maps defined by $u_t(x) = \Pi(u(x) + t\eta(x))$ $\forall \eta \in C_0^\infty(B, \mathbf{R}^k)$. Π is the nearest point projection of $\mathbf{R}^k$ into N.

There is another type of variation that one may consider. One takes $u_t = u \circ \varphi_t$ for φ_t a 1-parameter family of compactly supported C^1 diffeomorphisms of B with $\varphi_0 = \mathrm{Id}$. $E(u_t)$ is differentiable in t. If u is always critical for this type of variations and if u is harmonic, then u is called a stationary map.

So far not much is known about the regularity of weak harmonic maps. For $n = 2$ it is proved in [6] that a harmonic map with finite energy does not have isolated singularity. A theorem of [7] says that u has no interior singularity if u is stationary and $n = 2$.

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