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ON 4-DIMENSIONAL s-COBORDISMS

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0. Introduction

The idea that topological problems can be converted into questions of algebra and homotopy theory underlies much of modern higher-dimensional topology of manifolds. The s-cobordism theorem, also called the Barden-Mazur-Stallings theorem constitutes one of the basic building blocks of this approach. Let W be a compact $(n + 1)$-manifold with boundary the disjoint union of manifolds M_0 and M_1. Then this theorem asserts that for $n \geqslant 5$, W is diffeomorphic, piecewise linearly homeomorphic, or homeomorphic, depending on the category, to a product $M_0 \times [0,1]$, if and only if W has the homotopy type of this product and a certain algebraic invariant $\tau(W, M_0) \in Wh(\pi_1 W)$, the Whitehead torsion, vanishes. (See [12], [9] and [11] for the topological case.) This result, whose simply-connected version $\pi_1 W = \{e\}$ is just Smale's h-cobordism theorem, at least provides a direction of attack in the attempt to decide when two homotopy equivalent manifolds M_0 and M_1 are diffeomorphic—try to construct W or to measure the obstruction to doing so. Moreover, Freedman demonstrated that this s-cobordism theorem is valid for W a topological five-dimensional s-cobordism with fundamental group not too large, e.g. finite or polycyclic; i.e., he showed that such a five-dimensional W is homeomorphic to $M_0 \times [0,1]$ under the same hypothesis on the vanishing of the Whitehead torsion.

This paper provides a family of orientable counterexamples to the s-cobordism theorem in dimension four. (It seems to have been fairly widely understood (cf. [14]) that the realization of a certain non-orientable, non-smoothable normal invariant yields a nonorientable (and nonsmoothable) example.) Let Q_r be the quaternion group of order 2^{r+1};

$$Q_r = \{\, y, x \mid y^2 = x^{2r},\ yxy^{-1} = x^{-1} \}.$$

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