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MAPPINGS THAT MINIMIZE AREA IN THEIR HOMOTOPY CLASSES

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Let f: $M \to X$ be a continuous map from a compact connected oriented m-dimensional manifold M into a compact Riemannian manifold X. In this paper we consider the problem: does there exist a lipschitz map g: $M \to X$ that minimizes m-dimensional mapping area (or some other parametric elliptic functional) subject to the condition that g be homotopic to f? If so, what is the minimum area attained? And, if not, what is the infimum? It has long been known that in each *homology* class of X, there is an integral current that minimizes area (in that class). In this paper we show that, for $m \geqslant 3$, the homotopy problem reduces to the homology problem. For instance, if X is simply connected, the infimum area of mappings homotopic to f is equal to the minimum area among integral currents homologous to $f_\#([M])$ (where $[M] \in \mathscr{I}_m(M)$ is the m-dimensional integral cycle orienting M). Furthermore, if the current solution T is sufficiently regular, then the infimum is attained by a map whose image is the support of T together with a lower-dimensional singular set.

More generally, we allow M to be a compact manifold with (possibly empty) boundary. In this case, the homotopy problem is to minimize area among all maps g that are homotopic to f under homotopies H: $[0,1] \times M \to X$ that are fixed on ∂M (i.e., such that $H(t,x) = f(x)$ for $x \in \partial M$). Note that if $M = \mathbf{B}^m(0,1)$ and X is $\mathbf{R}^n$, this is the classical Plateau problem of minimizing area among maps g: $\mathbf{B}^m \to \mathbf{R}^n$ with boundary values $f|\partial\mathbf{B}^m$. Our main result is:

Theorem. *Suppose M is a compact connected oriented m-dimensional ($m \geqslant 3$) manifold with boundary, X is a Riemannian manifold (or more generally any local lipschitz neighborhood retract), and f: $M \to X$ is a lipschitz map. If X is simply connected (or if f_*: $\pi_1(M) \to \pi_1(X)$ is surjective), then*

$$\inf\{\text{Area}(g): g \text{ is homotopic to } f \text{ under a homotopy fixed on } \partial M\}$$
$$= \inf\{\text{Area}(T): T - f_*([M]) \text{ is an integral boundary in } X\}.$$

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