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THE TOTAL SQUARED CURVATURE OF CLOSED CURVES

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According to well-known arguments [1], a closed geodesic γ in a positively curved sphere M appears as a "minimax" critical point of the length functional $\mathfrak{L}$ and owes its existence to the higher homotopy of M. Alternatively, one could regard γ as a curve which yields a minimum, hence a *stable* equilibrium, for the total squared geodesic curvature functional $\mathfrak{F}$. Indeed, a (nongeodesic) circle on a round sphere is carried to a point under the flow of "$-\nabla\mathfrak{L}$", yet is carried to a nontrivial geodesic under the flow of "$-\nabla\mathfrak{F}$". One motivation for the present investigation is to gain insight into the question of what should happen under the latter flow to an *arbitrary* closed curve on the sphere or on another manifold M.

Thus, we undertake here to describe the set of critical points of $\mathfrak{F}$ defined on the regular closed curves in some concrete manifolds M, to examine the stability of these critical points, and to seek relationships among the various critical points.

Of course, the study of the total squared curvature of curves is not new. A classical *elastica*, following Daniel Bernoulli's model of an elastic rod in equilibrium, is a curve in R^2 or R^3 which is critical for $\mathfrak{F}$ defined on regular curves of a fixed length satisfying given first order boundary data (for historical references concerning the classical elastica, we refer the reader to the recent survey by Truesdell [11]).

Here we introduce the term *free elastica* to describe the critical points which result when the constraint on arclength is removed; these are among the Euclidean curves studied by Radon (whose work is described in Blaschke's *Vorlesungen über Differentialgeometrie*. I). Much more recently, Bryant and Griffiths [2], [5] considered the natural generalizations of the elastica and the free elastica to space forms, and showed how the general theory of exterior differential systems leads naturally to the integration of the equations for these curves.

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