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EXISTENCE OF CLOSED GEODESICS ON POSITIVELY CURVED MANIFOLDS

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In [8] we examined stability properties of closed geodesics whose existence can be obtained by elementary methods. In this paper we apply Lusternik-Schnirelmann theory to obtain the existence of several closed geodesics below certain length levels. We will then examine stability properties of these closed geodesics.

Let g_0 be the standard metric on S^n of constant curvature 1. Using perturbation methods it follows that any metric on S^n, $n \geqslant 2$, sufficiently C^2 close to g_0, has at least as many closed geodesics of length approximately 2π as a function on the Grassmannian $G_{2,n-1}$ of unoriented two-planes in R^{n+1} has critical points. This in turn can be estimated from below by the so-called cup length which for $G_{2,n-1}$ is $g(n) = 2n - s - 1$, where $0 \leqslant s = n - 2^k < 2^k$. Hence there are at least $g(n)$ short closed geodesics for metrics on S^n sufficiently C^2 close to g_0. Note that $\frac{1}{2}(3n - 1) \leqslant g(n) \leqslant 2n - 1$.

Theorem A. *Suppose that M is homeomorphic to S^n and that $1/4 \leqslant \delta \leqslant K \leqslant 1$, where K denotes the sectional curvature of M.*

(i) *There exist at least $g(n)$ closed geodesics without self-intersections and with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,] \subset [2\pi, 4\pi]$. If all closed geodesics of length $\leqslant 4\pi$ are nondegenerate (an open and dense condition on the set of metrics with respect to the C^2 topology), then there exist at least $n(n+1)/2$ closed geodesics without self-intersections and with lengths in $[2\pi, 2\pi/\sqrt{\delta}\,]$.*

(ii) *If the closed geodesics whose lengths lie in $[2\pi, 2\pi/\sqrt{\delta}\,]$ all have the same length l, then all geodesics are closed of length l. If the closed geodesics whose lengths lie in $[2\pi, 2\pi/\sqrt{\delta}\,]$ have only two different length values, then there exists a family of closed geodesics of equal length in $[2\pi, 2\pi/\sqrt{\delta}\,]$ such that every point of M lies in the image of some geodesic in the family.*

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