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CURVATURE OF AN ∞-DIMENSIONAL MANIFOLD RELATED TO HILL'S EQUATION

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1. Introduction

Let C_+^∞ be the space of positive infinitely differentiable functions e_0 of period 1 with $\int_0^1 e_0^2 = 1$, and let M be the class of real infinitely differentiable functions q of period 1 such that the corresponding Hill's operator $Q = -D^2 + q$ has ground state $\lambda_0 = 0$, where D signifies differentiation with regard to $0 \leqslant x < 1$. The map $C_+^\infty \to M$ defined by $e_0''/e_0 = q$ is $1:1$ and onto, the ground state of Q being necessarily simple; in particular, M comes in one simply-connected piece. The purpose of this note is to study the curvature of M considered as immersed in the space C_1^∞ of all real infinitely differentiable functions of period 1; evidently, it is a surface of codimension 1 defined by the single relation $\lambda_0 = 0$, and since the gradient of the latter is $\nabla\lambda_0 = e_0^2 \neq 0$, M sits smoothly in C_1^∞.

The curvatures of 2-dimensional slices of M are found to be positive, the principal curvatures being proportional to the reciprocals of the excited periodic eigenvalues $0 < \bar{\lambda}_j$ ($j = 1, 2, 3, \cdots$) of the so-called *allied operator* $\bar{Q}$. The latter is the Hill's operator with ground state proportional to $e_0^{3/2}$ relative to the scale $d\bar{x} = (\int_0^1 e_0)^{-1} e_0\,dx$. The *maximal curvature* of a 2-dimensional slice is

$$m = 4\left(\int_0^1 e_0\right)^4 \left(\int_0^1 e_0^4\right)^{-1} \times (\bar{\lambda}_1 \bar{\lambda}_2)^{-1},$$

while the *total curvature* is

$$k = 4\left(\int_0^1 e_0\right)^4 \left(\int_0^1 e_0^4\right)^{-1} \times \sum_{1 \leqslant i < j} (\bar{\lambda}_i \bar{\lambda}_j)^{-1}.$$

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