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# A TOPOLOGICAL GAUSS-BONNET THEOREM

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## 0. Introduction

The generalized Gauss-Bonnet theorem of Allendoerfer-Weil [1] and Chern [2] has played an important role in the development of the relationship between modern differential geometry and algebraic topology, providing in particular one of the primary stimuli for the theory of characteristic classes. There are now a number of proofs in the literature, from the quite sophisticated (deducing it as a special case of the Atiyah-Singer index theorem for example) to the relatively elementary and straightforward. (For a particularly elegant example of the latter see [7, Appendix C].) In general these previous proofs have a definite cohomological flavor and invoke explicit appeals to general vector bundle or principal bundle theory. In view of the above historical fact this is perhaps natural, and yet from another point of view it is somewhat anomalous. For the theorem states the equality of two quantities:

$$(2\pi)^{-k}\int_M K^{(n)}d\mu = \chi(M) .$$

Here $M$ is any closed ($=$ compact, without boundary), smooth ($= C^\infty$) Riemannian manifold of even dimension $n = 2k$, $K^{(n)}$ is a certain "natural" real valued function on $M$ (which in local coordinates is a somewhat complicated but quite explicit rational function of the components of the metric tensor and its partial derivatives of order two or less), $\mu$ is the Riemannian measure, and $\chi(M)$ is the Euler characteristic of $M$. There is nothing fundamentally "cohomological" on either side of this identity. True, one tends to think of $\chi(M)$ as the alternating sum of the betti numbers, but equally well and more geometrically it is the self intersection number of the diagonal in $M \times M$ or equivalently the algebraic number of zeros of a generic vector field. Indeed $\chi(M)$ is perhaps the most primitive topological invariant of $M$ beyond the number of connected components; the fact that $\Sigma(-1)^k n_k$ (where $n_k$ is the number of faces of dimension $k$ in a cellular decomposition of a polyhedron $P$) is a combinatorial invariant $\chi(P)$ goes back two hundred years before the development of homology theory. And on the left we are *really* integrating a function with respect

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