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CURVATURES OF COMPLEX SUBMANIFOLDS OF C^n

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0. Introduction

Complex submanifolds M^n of a complex N-space C^N from the viewpoint of hermitian geometry are distinguished by

(a) the existence of N holomorphic imbedding functions $f_1, f_2, \cdots, f_N$ so that the kähler form is of the form $i\partial\bar{\partial}(\sum |f_i|^2)$, and as a consequence

(b) the imbedding is minimal in the sense of riemannian geometry, and all the holomorphic sectional curvatures are nonpositive. In [2] Bochner demonstrated that the Poincaré metric of constant negative curvature on the unit disc cannot be holomorphically imbedded in C^N even locally. It seems therefore reasonable to pose the following

Question. Does there exist a complete complex submanifold M^n of C^N with holomorphic sectional curvature bounded away from zero?

In this paper we discuss partial results to this question. To begin with, we show in § 1 that a negative answer to this question would imply that there is no bounded complete complex submanifold of C^N. In § 2, utilizing an elementary observation on the Gauss map we answer the question in the negative for hypersurfaces, and in § 3 we show that it suffices to consider the question for holomorphic curves ($n = 1$).

In § 4 we recall the higher order curvature functions introduced by Calabi and show that two such functions are enough to determine a holomorphic curve uniquely up to a rigid motion in C^N, and thus providing a justification for a generalization of the theorem in § 2, in terms of the higher order curvature functions. In § 5, applying the method of extremal length we derive a criterion, which involves the curvature behavior at infinity of a simply connected metric riemann surface M for it to be conformally equivalent to the disc. It is subsequently used to sharpen the result in § 2.

The last section contains curvature estimate for a piece of curve in C^2 which is a graph over a domain in C.

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