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HOMOGENEOUS CONVEX DOMAINS OF NEGATIVE SECTIONAL CURVATURE

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Let Ω be an affine homogeneous convex domain in a finite dimensional real vector space V, not containing any full straight line. Then we know that Ω admits an invariant volume element

$$v = K dx^1 \wedge \cdots \wedge dx^n$$

and that the *canonical bilinear form*

$$D\alpha = \sum_{i,j} \frac{\partial^2 \log K}{\partial x^i \partial x^j} dx^i dx^j$$

defines an invariant Riemannian metric on Ω, [2], [6]. In this note we prove the following theorem.

Theorem. *An affine homogeneous convex domain Ω not containing any full straight line has negative sectional curvature with respect to $D\alpha$ if and only if Ω is the interior of a paraboloid*:

$$y^0 - \frac{1}{2} \sum_{i=1}^{n-1} (y^i)^2 > -1 ,$$

where $\{y^0, y^1, \cdots, y^{n-1}\}$ is an affine coordinate system of V.

We first recall the construction of clans from homogeneous convex domains, [6]. In the following we assume that a homogeneous convex domain Ω contains the zero vector 0. Let G be a connected triangular affine Lie group which acts simply transitively on Ω, and let $\mathfrak{g}$ be the affine Lie algebra corresponding to G. For $X \in \mathfrak{g}$, we denote by $f(X)$, $q(X)$ the linear part and the translation vector of X respectively. Since q is a linear isomorphism of $\mathfrak{g}$ onto V, for each $x \in V$ there exists a unique $X_x \in \mathfrak{g}$ such that $q(X_x) = x$. We define an operation of multiplication in V by the formula

$$(1) \qquad x \cdot y = f(X_x) y \qquad \text{for } x, y \in V .$$

Then we have

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