J. DIFFERENTIAL GEOMETRY
12 (1977) 87-100

APPROXIMATE EIGENFUNCTIONS OF THE LAPLACIAN

J. V. RALSTON

Introduction

Let M be a compact orientable $(n + 1)$-dimensional riemannian manifold, and let Γ be a closed geodesic on M. We say Γ is stable if the Poincaré map associated with Γ (this is defined in § 1) splits into a direct sum of rotations through distinct angles $\theta_1, \cdots, \theta_n$, $0 < \theta_i < 2\pi$, $\theta_i \neq 2\pi - \theta_j$ for all i, j. Let Δ denote the Laplace-Beltrami operator on M. Guillemin and Weinstein [5] have recently proved the following.

Theorem I. *If there is a stable closed geodesic on M of length L, then, given any multi-index α, there are at least two eigenvalues λ_m of Δ, counted by multiplicity, satisfying $\sqrt{-\lambda_m} = k_m + 0(m^{-\frac{1}{2}})$, where $k_m = L^{-1}(2\pi m + (\alpha_1 + \frac{1}{2})\theta_1 + \cdots + (\alpha_n + \frac{1}{2})\theta_n + \pi p_0)$. Here $p_0 = 0$ or 1 and is determined by the behavior of the Jacobi fields along Γ.*

Since a rotation through an angle θ can turn into a rotation through $2\pi - \theta$ if one changes bases, there is a technical condition that determines which of these rotations one chooses in selecting $\theta_1, \cdots, \theta_n$ in Theorem I. We omit this here; see § 2.

Guillemin and Weinstein's proof of Theorem I is based on the construction of an isometry from $L^2(S^1)$ to $L^2(M)$ that approximately intertwines $d^2/d\theta^2$ and Δ. The isometry is a Fourier integral operator of a new type developed by Guillemin in [4].

Our objective here is to prove Theorem I and its analogue for nonorientable M by constructing approximate solutions u_m to the equations $(\Delta + k_m^2)u = 0$. The functions u_m are probably very close to the image of $\{e^{im\theta}\}_{m=1}^{\infty}$ under the isometry used by Guillemin and Weinstein. However, we construct them by beginning with the ansatz of geometrical optics and using a complex phase function ψ with $\operatorname{Im} \psi > 0$ off Γ and $\operatorname{Im} \psi = 0$ on Γ. The resulting u_m are very small outside a tube around Γ with radius $0(m^{-\frac{1}{2}})$. The construction is quite explicit, expressing the u_m in terms of the Jacobi fields along Γ.

Our approach is derived from the work of Babich and Lazutkin [2] who used a similar method to prove Theorem I in the case $n = 1$. We found the idea which enabled us to carry out the construction for general n in the paper [6] of Hörmander.

Received March 22, 1975.