J. DIFFERENTIAL GEOMETRY
11 (1976) 499-520

TOPOLOGY OF THE COMPLEX VARIETIES $A_s^{(n)}$

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1. Introduction

Define, for $s \leq [n/2]$,

$\tilde{V}_{n,2s}$: manifold of ordered $2s$-tuplets of linearly independent vectors in Euclidean n-space R^n,

$\tilde{A}_s^{(n)}$: space of 2-forms in R^n of rank $2s$,

$\tilde{f}_s^{(n)}: \tilde{V}_{n,2s} \to \tilde{A}_s^{(n)}$: map given by

$$\tilde{f}_s^{(n)}(y_1, \cdots, y_{2s}) = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s} ,$$

$V_{n,2s}$: Stiefel manifold of orthonormal $2s$-frames in R^n,

$A_s^{(n)} = \tilde{f}_s^{(n)}(V_{n,2s})$: subspace of $\tilde{A}_s^{(n)}$ of "normalized" 2-forms in R^n of rank $2s$,

$f_s^{(n)}: V_{n,2s} \to A_s^{(n)}$: the restriction of $\tilde{f}_s^{(n)}$ to $V_{n,2s}$.

It was proved in [4] that the maps $\tilde{f}_s^{(n)}$ and $f_s^{(n)}$ induce the principal $Sp(s; R)$- and $U(s)$-bundles respectively, and that $A_s^{(n)}$ is a strong deformation retract of $\tilde{A}_s^{(n)}$.

One may, equivalently, define $A_s^{(n)}$ as the space of normalized complex s-substructures of R^n, i.e., pairs (p, J) where p is a $2s$-plane in R^n and J is a normalized complex structure on p ($J \in O(p)$, $J^2 = -1$).

To see the equivalence, let $w \in A_s^{(n)}$. Then $w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$ for an orthonormal $2s$-frame $y = (y_1, \cdots, y_{2s})$. Let p be the $2s$-plane spanned by y. For $x \in p$, let $d_x: p \to \Lambda^2 p$ be forming wedge products with x, i.e., $d_x(z) = x \wedge z$, and $\delta_x: \Lambda^2 p \to p$ be its "adjoint". Define a linear transformation J on p by $J(x) = \delta_x(w)$, $x \in p$. Then $J(y_i) = y_{i+s}$ and $J(y_{i+s}) = -y_i$, $1 \leq i \leq s$. Thus $J \in O(p)$, $J^2 = -1$. Conversely, a normalized complex s-substructure J, $J \in O(p)$, $J^2 = -1$, can be represented by the matrix $\begin{bmatrix} 0 & -I_s \\ I_s & 0 \end{bmatrix}$ relative to some orthonormal $2s$-frame $y = (y_1, \cdots, y_{2s})$ on p. Hence J corresponds to $w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$ in $A_s^{(n)}$.

It follows from either definition that $A_s^{(n)} = SO(n)/U(s) \times SO(n-2s)$ for $s < n/2$, $A_s^{(2s)} = O(2s)/U(s) = I_s \cup I_s'$ where $I_s = SO(2s)/U(s)$, $A_1^{(n)} = \tilde{G}_{n,2} = Q_{n-2}(C)$ where $\tilde{G}_{n,2}$ is the oriented 2-planes in R^n, and $Q_{n-2}(C)$ is the complex quadric of dimension $n-2$.

The spaces $A_s^{(n)}$ appear as "fibres" in global obstruction problems involving

Received May 1, 1973, and, in revised form, January 16, 1974.