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CONDITION (C) FOR THE ENERGY INTEGRAL ON CERTAIN PATH SPACES AND APPLICATIONS TO THE THEORY OF GEODESICS

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Introduction

Let M be a complete connected Riemannian manifold, and $L_1^2(I, M)$ the Hilbert manifold of absolutely continuous maps from the unite interval $I = [0, 1]$ to M with square integrable derivative. See, e.g., Eells [4] for the manifold structure on $L_1^2(I, M)$, or Karcher [9] and Palais [16] for analogous spaces. There are various interesting submanifolds of $L_1^2(I, M)$ related to the study of different kinds of geodesics on M, which appear as critical points for the energy integral on the submanifolds.

This paper is divided into three sections. In the first two sections we point out some interesting submanifolds of $L_1^2(I, M)$ and their related geodesics on M, and study to which extent the energy integral satisfies Condition (C) of Palais and Smale (a necessary condition for making critical point theory like Morse theory and Lusternik-Schnirelmann theory on infinite dimensional manifolds). Our first result was a generalization of those obtained by McAlpin or Karcher [9] and Palais [16]. However Eliasson has recently obtained a general result on Condition (C) based on the notion of weak submanifolds and local coercive properties of the involved function [5]. Conversations with Eliasson made it clear that his results applies to our case, so that Theorem 2.4 now in some sense is the best possible result on Condition (C) for the energy function on path spaces. The author is indebted to Eliasson for pointing out this to him. Immediate applications of Theorem 2.4 are made to geodesics between submanifolds of M and to geodesics invariant under an isometry without fixed points; by invariant we mean that the geodesic is mapped onto itself with the direction of speed preserved. In the last section we apply the results of the first two sectionsto get existence thorems for geodesics on a compact manifold invariant under a given isometry. Our main results in the last section are contained in

Theorem. *Let M be a compact Riemannian manifold, and $A: M \to M$ an isometry on M.*

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