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BOUNDED SETS AND FINSLER STRUCTURES FOR MANIFOLDS OF MAPS

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Abstract infinite dimensional manifolds modelled on Banach spaces lack much of the topological structure of both finite dimensional manifolds and their linear Banach space models. In this paper we show that certain manifolds of maps between finite dimensional manifolds, or more generally manifolds of sections of a finite dimensional fiber bundle, have an additional natural structure of sets which we call "intrinsically bounded" which have many of the properties of bounded sets in the linear model. Theorem 1 shows that these sets can be characterized in several different ways. Our results are specifically stated for the Sobolev manifolds $L_k^p(E)$ where E is a fiber bundle over the compact manifold M of dimension less than pk. We also construct a canonical Finsler structure for $L_k^p(E)$ from geometrical structure on E, and find Finsler structures which have intrinsically bounded sets as their bounded sets. The discussion of Finsler structures will be helpful in freeing the use of condition (C) of Palais and Smale in the calculus of variations from the unnaturally arbitrary choice of Finsler structure on the manifolds of maps.

Many of the definitions and formal statements of theorems are due to R. S. Palais. Construction and properties of a weak topology have been obtained by D. Graff. U. Koschorke has developed a more abstract theory for Banach manifolds with specified atlases [4]. J. Dowling has shown that minimizing geodesics exist for the Finsler structures discussed in the second section [1]. A good many of the results of this section were obtained at the same time by H. Eliasson [3] but without the construction of bounded sets. Eliasson also gives the construction in Appendix II in a different style [2].

All manifolds and maps are C^∞ unless otherwise stated. The results are given for sections of a fiber bundle E over a compact base manifold M, with or without bundary, where the fibers are finite dimensional manifolds without boundary. The reader is periodically reminded that $E = M \times N$ is an important case where the sections are merely maps between M and N. Most of the necessary inequalities are of the same general type, which is discussed in Appendix I, and they are therefore not explained in the paper. The construction of the covariant derivatives which are used has been left to Appendix II.

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