J. DIFFERENTIAL GEOMETRY
6 (1971) 267-270

THE CONVEX HULL PROPERTY OF IMMERSED MANIFOLDS

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Let M_0 be an m-dimensional differentiable manifold, and let M be defined by an immersion $f: M_0 \to \boldsymbol{R}^n$. We shall say that M has the *convex hull property* if, for every domain D on M_0 such that f maps D into a bounded set in $\boldsymbol{R}^n$, the image of D lies in the convex hull of its boundary values. It is well known that minimal submanifolds of $\boldsymbol{R}^n$ have this property. The usual proof of this uses the fact that the coordinate functions in $\boldsymbol{R}^n$ are harmonic functions on M if M is minimal. (See, for example, Lemma 7.1 in [1] for the case $m = 2$, n arbitrary. We may note that the proof there is not quite correct as it stands, since the assumption that the image lies in a bounded set is implicitly used but not stated.)

Our purpose here is to give a simple geometric condition characterizing those manifolds having the convex hull property.

Theorem. *A manifold M has the convex hull property if and only if, at each point of M, there does not exist any normal direction with respect to which all normal curvatures of M are positive.*

Remarks. 1. If the normal curvatures with respect to a normal N are all negative, then those with respect to $-N$ are all positive. Thus the condition of the theorem is equivalent to the property that, for each normal N, the range of values of the normal curvatures with respect to N includes zero. If we denote the principal curvatures with respect to N in decreasing order by

$$k_1(N) \geq k_2(N) \geq \cdots \geq k_m(N) ,$$

then another equivalent formulation is

$$k_1(N)k_m(N) \leq 0 \qquad \text{for every normal } N . \tag{1}$$

Thus for surfaces in $\boldsymbol{R}^3$, the condition of the theorem is simply that the Gauss curvature be everywhere nonpositive.

2. If M is minimal, then at each point,

$$k_1(N) + \cdots + k_m(N) = 0 \qquad \text{for every normal } N . \tag{2}$$

Received April 24, 1971. This work was partially supported by a grant from NSF.