J. DIFFERENTIAL GEOMETRY
6 (1971) 175-192

A GENERALIZATION OF THE ISOPERIMETRIC INEQUALITY

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1. For a simple closed plane curve of length L bounding an area A the classical isoperimetric inequality asserts that

$$L^2 - 4\pi A \geq 0 ,$$

with equality holding only for a circle. We show here that this inequality remains true for non-simple closed curves where in place of A we take the sum of the areas into which the curve divides the plane, each weighted with the square of the winding number, i.e.,

$$L^2 - 4\pi \int_{E^2} w^2 dA \geq 0 ,$$

where, for $p \in E^2$, $w(p)$ is the winding number of p with respect to the curve. Equality holds if and only if the curve is a circle, or a circle traversed several times or several coincident circles each traversed in the same direction any number of times. Note that this implies that

$$L^2 - 4\pi \int_{E^2} |w|^p dA \geq 0$$

for any $0 < p \leq 2$ and that 2 is here the best possible power.

This may all be generalized to arbitrary dimension and codimension. For the case of closed space curves let G denote the space of lines in E^3 (parallel lines are not identified) and let dG denote its invariant measure [1], [7]. Then

$$L^2 - 4 \int_G \lambda^2 dG \geq 0 ,$$

where $\lambda(l)$ denotes the linking number of $l \in G$ with the curve. Equality holds

Received February 14, 1970. The work of the first author was supported by the National Science Foundation under Grant GP-7610, and that of the second author by the N.S.F. under Grant GP-5760 and the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).