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A REMARK ON HOLMGREN'S UNIQUENESS THEOREM

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In a recent note Bony [1] has given a remarkable improvement of Holmgren's uniqueness theorem. The result is as follows. Let $P(x, D)$ be a differential operator with analytic coefficients in a neighborhood X of a point $x_0 \in \boldsymbol{R}^n$, and denote the principal symbol by $P_m(x, \xi)$ where $x \in X$ and $\xi \in \boldsymbol{R}^n$. Let $u \in \mathscr{D}'(X)$ be a solution of the equation $P(x, D)u = 0$ vanishing when $\varphi(x) > \varphi(x_0)$, $x \in X$, where $\varphi \in C^1(X)$ and $N_0 = \operatorname{grad} \varphi(x_0) \neq 0$. Holmgren's uniqueness theorem then states that u must vanish in a neighborhood of x_0 if $P_m(x_0, N_0) \neq 0$. (Schapira [4] has proved that this remains true for hyperfunction solutions.) Bony [1] introduced the smallest ideal $I(P)$ in $C^\infty(X \times (\boldsymbol{R}^n \backslash 0), \boldsymbol{R})$ such that

(i) $Q \in I(P)$ if $Q(x, \xi)$ is positively homogeneous with respect to ξ and vanishes for all $(x, \xi) \in X \times (\boldsymbol{R}^n \backslash 0)$ with $P_m(x, \xi) = 0$,

(ii) $Q_1, Q_2 \in I(P)$ implies $\{Q_1, Q_2\} \in I(P)$ if

$$\{Q_1, Q_2\} = \sum (\partial Q_1/\partial \xi_j \, \partial Q_2/\partial x_j - \partial Q_1/\partial x_j \, \partial Q_2/\partial \xi_j)$$

is the Poisson bracket of Q_1 and Q_2.

Bony's result is that $u = 0$ in a neighborhood of x_0 unless all $Q \in I(P)$ vanish at (x_0, N_0). The idea of the proof is that if the boundary of supp u is smooth, then the functions satisfying condition (i) must vanish on the normal bundle by Holmgren's uniqueness theorem, so the classical integration theory for first order equations shows that the repeated Poisson brackets of such functions must vanish too. The surprising point in the argument of Bony is that although the support of u may be an arbitrary closed set a priori, one can find sufficiently good parametrizations of large parts of a generalized normal bundle in order to carry through this argument. Although very ingenious the proof seems slightly artificial in that it forces one to introduce highly irregular objects into consideration. We shall here give an alternative more elementary proof based on deformations of smooth surfaces passing through the point x_0 such that repeated use of Holmgren's uniqueness theorem gives the desired result. This is analogous to the proof of Theorem 5.3.2 in Hörmander [2], which gives another variant of Holmgren's uniqueness theorem for the case where P_m is real, $\varphi \in C^2$, $P_m(x_0, N_0) = 0$ but the second order derivative of φ along the bicharacteristic curve with initial data (x_0, N_0) is positive. Indeed our proof here will also give

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