THE NORMAL SINGULARITIES OF A SUBMANIFOLD

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0. Introduction

Let $\mathbf{R}^n$ be furnished with its Euclidean bilinear scalar product $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$; $(x, x') \mapsto x \cdot x'$ and associated positive-definite quadratic form $\mathbf{R}^n \to \mathbf{R}$; $x \mapsto x^{(2)} = x \cdot x$, and let M be an m-dimensional smooth (=sufficiently differentiable) submanifold of $\mathbf{R}^n$. Locally M may be represented parametrically as the image of a smooth embedding $g: \mathbf{R}^m \rightarrowtail \mathbf{R}^n$. (The tail on the arrow denotes that the domain of g is not necessarily the whole of $\mathbf{R}^m$.) Consider the map

$$\phi: M \times \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n; \qquad (w, x) \mapsto ((x - w)^{(2)}, x) ,$$

represented locally by the map

$$f: \mathbf{R}^m \times \mathbf{R}^n \rightarrowtail \mathbf{R} \times \mathbf{R}^n; \qquad (t, x) \mapsto ((x - g(t))^{(2)}, x) .$$

Our purpose is to describe the Whitney-Thom generic singularities $\Sigma^I \phi$ of ϕ ([1] and also [19]), at least for small values of m and n. For the smallest values this turns out to be a re-exposition from a fresh point of view of some well-known facts of elementary differential geometry [21]. The inspiration for studying the map ϕ is a remark of **R**. Thom in his book [27, Chapter 4], where he justifies the use of the word 'umbilic' to describe certain of the elementary catastrophes. (See § 13 below.)

The main results which seem to be new, at least in detail, are in § 9, on what happens at an umbilic of a generic surface in $\mathbf{R}^3$ and in § 11, on what happens at a parabolic umbilic of a generic surface in $\mathbf{R}^4$. Umbilics have recently been studied by Feldman [7], [8] from a somewhat different point of view. The classical references are Darboux [5], Picard [18] and Gullstrand [11]. In [13], Hartman and Wintner made some corrections to Picard. The author is grateful to many people, especially Professor C. B. Allendoerfer and Professor W. L. Edge, for acquainting him with classical papers he had never read, and for reminding him of things he once knew but had forgotten. The author is also grateful to Professor R. Thom for hinting that there was more to umbilics than Thom actually stated in the first draft of his book.