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A CHARACTERIZATION OF A STANDARD TORUS IN E^3

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0. Introduction

Let M be a two dimensional, connected, complete and orientable Riemannian manifold of class C^∞, and $\iota\colon M \to E^3$ be an isometric immersion of M into a Euclidean three space. The purpose of the present paper is to find some conditions for M to be congruent to a standard torus in E^3; by a standard torus in E^3 we mean a surface of revolution defined by

$$x = (a + b\cos u)\cos v\,, \qquad y = (a + b\cos u)\sin v\,, \qquad z = b\sin u\,,$$
$$a > b > 0, 0 \leq u < 2\pi, 0 \leq v < 2\pi\,,$$

which we shall denote by $T(a, b)$. One of the properties of a standard torus is that one of its principal curvatures is constant everywhere. There are a lot of classes of surfaces with such property, for example, sphere, right circular cylinder, standard torus, etc.. A characterization of a standard torus seems to be more complicated than those of a sphere or right circular cylinder under the condition that one of the principal curvatures is constant everywhere, since a standard torus has non-constant mean curvature and its Gaussian curvature changes sign. The authors were inspired on this subject by one of the problems stated by Willmore in [4], and were informed of this problem by Professor M. Obata.

Problem (*Willmore* [4]). Let $\iota\colon M \to E^3$ be an imbedding of a compact and orientable manifold M of genus 1 into E^3, and H be the mean curvature of $\iota(M)$ with respect to the induced metric from E^3. Then, does the following equality hold?

$$\inf_{\iota} \int_{\iota(M)} H^2 dA = 2\pi^2\,,$$

where dA denotes the area element of M and ι ranges over all imbeddings of M into E^3.

The main theorem of the present paper gives a partial solution to the above problem, and can be stated as follows:

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