J. DIFFERENTIAL GEOMETRY
1 (1967) 355-368

ALMOST COMPLEX STRUCTURES ON TENSOR BUNDLES

A. J. LEDGER & K. YANO

1. Introduction

It is well known that the tangent bundle of a C^∞ manifold M admits an almost complex structure if M admits an affine connection [1], [5] or an almost complex structure [7], [8]. The main purpose of this paper is to investigate a similar problem for tensor bundles T_s^rM. We prove that if a Riemannian manifold M admits an almost complex structure then so does T_s^rM provided $r + s$ is odd. If $r + s$ is even a further condition is required on M. The proofs depend on some generalizations of the notions of lifting vector fields and derivations on M, which were defined previously only for tangent bundles and cotangent bundles [4], [7], [8], [9], [10].

2. Notations and definitions

(i) M is a C^∞ paracompact manifold of finite dimension n.

(ii) $F(M)$ is the ring of real-valued C^∞ functions on M.

(iii) For $r + s > 0$, T_s^rM is the bundle over M of tensors of type (r, s), contravariant of order r and covariant of order s. π is the projection of T_s^rM onto M. We write $T_0^rM = T^rM$, $T_s^0M = T_sM$.

(iv) $\mathscr{T}_s^r(M)$ is the module over $F(M)$ of C^∞ tensor fields of type (r, s). We write $\mathscr{T}_0^r(M) = \mathscr{T}^r(M)$, $\mathscr{T}_s^0(M) = \mathscr{T}_s(M)$, and $\mathscr{T}_0^0(M) = F(M)$. $\mathscr{T}(M)$ is the direct sum $\sum_{r,s} \mathscr{T}_s^r(M)$. T_p is the value at $p \in M$ of a tensor field T on M, and $\mathscr{T}_s^r(p)$ is the vector space of tensors of type (r, s) at p.

(v) Let $S \in \mathscr{T}_r^s(p)$ and $T \in \mathscr{T}_s^r(p)$. Then the real number $S(T) = T(S)$ is defined, in the usual way, by contraction. It follows that if $S \in \mathscr{T}_r^s(M)$ then S is a differentiable function on T_s^rM.

(vi) A map $D: \mathscr{T}(M) \to \mathscr{T}(M)$ is a derivation on M if
   (a) D is linear with respect to constant coefficients,
   (b) for all r, s, $D\mathscr{T}_s^r(M) \subset \mathscr{T}_s^r(M)$,
   (c) for all C^∞ tensor fields T_1 and T_2 on M,

$$D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes DT_2 ,$$

Communicated August 15, 1967.