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THE THEORY OF QUASI-SASAKIAN STRUCTURES

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Introduction

On a contact manifold of dimension $2n + 1$ there exists, by definition, a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. An almost contact manifold also carries a 1-form η but it is not necessarily of maximal rank. The purpose of this paper is to explore the meaning of the rank of η. To this end, we initiate the study of normal almost contact metric manifolds with closed fundamental 2-form Φ. Such manifolds will be called quasi-Sasakian manifolds.

§1 presents the basic definitions and some results from the theory of almost contact structures. Beginning with §2 we develop the theory of quasi-Sasakian structures. In §2 a large class of examples is given and in §3 we discuss the meaning of the rank of η. The result is that if η has rank $2p + 1$ and the determined almost product structure is integrable then the manifold is locally the product of a Sasakian (normal contact metric) manifold and a Kaehler manifold. That is to say, η having rank $2p + 1$ means, loosely speaking, that the space is split locally into a Sasakian piece where $\eta \wedge (d\eta)^p \neq 0$ and a Kaehler piece whose fundamental 2-form is $\Phi - d\eta$ properly restricted. §4 gives some geometric results on quasi-Sasakian manifolds and §5 characterizes the case where $d\eta = 0$, the latter characterization being necessary in the study of the topology of cosymplectic manifolds [1], [2].

1. Almost contact manifolds

All manifolds considered will be C^∞ and connected. A superscript will denote the dimension of the manifold, for example M^{2n+1}, and $\mathscr{E}^{2n+1}$ will denote the module of vector fields over M^{2n+1}. When we speak of an almost contact manifold, quasi-Sasakian manifold, etc., we mean the manifold together with the corresponding structure.

A $(2n + 1)$-dimensional manifold carrying a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ is said to have a *contact structure* with η as its *contact form*. On the other hand, a manifold M^{2n+1} has an *almost contact structure* (ϕ, ξ, η)

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