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DISCRIMINANT OF THETA DIVISORS AND QUILLEN METRICS

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Abstract

We show that analytic torsion of smooth theta divisor is represented by a Siegel modular form characterizing the Andreotti-Mayer locus.

1. Introduction

In the theory of modular forms of one variable, the unique cusp form of weight 12 called Jacobi's Δ-function:

$$\Delta(\tau) = q \prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = \exp(2\pi i\tau) \tag{1.1}$$

is one of the most important objects. There are several view points to see it. From an algebraic view point, it is the discriminant of elliptic curves. To be precise, let $E_\tau := \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ ($\tau \in \mathbb{H}$) be an elliptic curve and take its Weierstrass model: $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$. Jacobi discovered the following formula:

$$g_2(\tau)^3 - 27g_3(\tau)^2 = (2\pi)^{12}\Delta(\tau). \tag{1.2}$$

Namely $\Delta(\tau)$ is the discriminant of the polynomial $4x^3 - g_2(\tau)x - g_3(\tau)$.

From an analytic view point, $\Delta(\tau)$ is essentially the Ray-Singer analytic torsion. Equipped with the Kähler metric $g_\tau = (\mathrm{Im}\tau)^{-1}|dz|^2$, analytic torsion of (the trivial line bundle on) E_τ is, by definition (Definition 2.1), $\tau(E_\tau) = \exp(\zeta_\tau'(0))$ where

$$\zeta_\tau(s) = (2\pi)^{-2s} \sum_{(m,n)\neq(0,0)} \frac{(\mathrm{Im}\tau)^s}{|m+n\tau|^{2s}} \tag{1.3}$$

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