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FUNCTORIAL RELATIONSHIPS BETWEEN $QH^*(G/B)$ AND $QH^*(G/P)$

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Abstract

We give a natural filtration $\mathcal{F}$ on $QH^*(G/B)$, which respects the quantum product structure. Its associated graded algebra $Gr^{\mathcal{F}}(QH^*(G/B))$ is isomorphic to the tensor product of $QH^*(G/P)$ and a corresponding graded algebra of $QH^*(P/B)$ after localization. When the quantum parameter goes to zero, this specializes to the filtration on $H^*(G/B)$ from the Leray spectral sequence associated to the fibration $P/B \to G/B \longrightarrow G/P$.

1. Introduction

Let G be a simply connected complex simple Lie group, B be a Borel subgroup, and $P \supset B$ be a parabolic subgroup of G. The natural fibration $P/B \to G/B \longrightarrow G/P$ of homogeneous varieties gives rise to a $\mathbb{Z}^2$-filtration $\mathcal{F}$ on $H^*(G/B)$ over $\mathbb{Q}$ (or $\mathbb{C}$) such that $Gr^{\mathcal{F}}(H^*(G/B)) \cong H^*(P/B) \otimes H^*(G/P)$ as graded algebras by the Leray-Serre spectral sequence. Given another parabolic subgroup P' with $B \subset P' \subset P$, we obtain the corresponding natural fibration $P'/B \to P/B \longrightarrow P/P'$. Combining it with the former one, we obtain a $\mathbb{Z}^3$-filtration on $H^*(G/B)$. We can continue this procedure to obtain a (maximal) $\mathbb{Z}^{r+1}$-filtration.

In the present paper, we study the small quantum cohomology rings $QH^*(G/P)$'s of homogeneous varieties G/P's, which are deformations of the ring structures on $H^*(G/P)$'s by incorporating genus zero 3-pointed Gromov-Witten invariants of G/P's into the cup product. We show the "*functorial relationships*" between $QH^*(G/B)$ and $QH^*(G/P)$ in the sense that the $\mathbb{Z}^{r+1}$-filtration on $H^*(G/B)$ can be generalized to give a $\mathbb{Z}^{r+1}$-filtration on $QH^*(G/B)$ and there exist canonical maps between quantum cohomologies, in analog with the classical ones. We begin with a toy example to illustrate our results.

Example 1.1. When $G = SL(3, \mathbb{C})$, $G/B = \{V_1 \leqslant V_2 \leqslant \mathbb{C}^3 \mid \dim_{\mathbb{C}} V_i = 1, i = 1, 2\} =: F\ell_3$ is a complete flag variety. Given a maximal parabolic subgroup $P \supset B$, we have $P/B = \mathbb{P}^1$ and $G/P = \mathbb{P}^2$ together with a natural fibration $\mathbb{P}^1 \overset{i}{\hookrightarrow} F\ell_3 \overset{\pi}{\longrightarrow} \mathbb{P}^2$. The quantum cohomology ring

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