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# CONSTRAINED WILLMORE TORI IN THE 4–SPHERE

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## Abstract

We prove that a constrained Willmore immersion of a 2–torus into the conformal 4–sphere $S^4$ is of "finite type", that is, has a spectral curve of finite genus, or of "holomorphic type" which means that it is super conformal or Euclidean minimal with planar ends in $\mathbb{R}^4 \cong S^4\backslash\{\infty\}$ for some point $\infty \in S^4$ at infinity. This implies that all constrained Willmore tori in $S^4$ can be constructed rather explicitly by methods of complex algebraic geometry. The proof uses quaternionic holomorphic geometry in combination with integrable systems methods similar to those of Hitchin's approach [**19**] to the study of harmonic tori in $S^3$.

## 1. Introduction

A conformal immersion of a Riemann surface is called a *constrained Willmore surface* if it is a critical point of the Willmore functional $\mathcal{W} = \int_M |\mathring{\mathrm{I\!I}}|^2 dA$ (with $\mathring{\mathrm{I\!I}}$ denoting the trace free second fundamental form) under compactly supported infinitesimal conformal variations, see [**23**, **27**, **8**, **5**]. The notion of constrained Willmore surfaces generalizes that of Willmore surfaces which are the critical points of $\mathcal{W}$ under all compactly supported variations. Because both the functional and the constraint of the above variational problem are conformally invariant, the property of being constrained Willmore depends only on the conformal class of the metric on the target space. This suggests an investigation within a Möbius geometric framework like the quaternionic projective model of the conformal 4–sphere used throughout the paper.

The space form geometries of dimension 3 and 4 occur in our setting as subgeometries of 4–dimensional Möbius geometry and provide several classes of examples of constrained Willmore surfaces, including constant mean curvature (CMC) surfaces in 3–dimensional space forms and minimal surfaces in 4–dimensional space forms. See [**5**] for an introduction to constrained Willmore surfaces including a derivation of the Euler–Lagrange equation for compact constrained Willmore surfaces.

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