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ON HYPERBOLIC GROUPS WITH SPHERES AS BOUNDARY

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Dedicated to Steve Ferry on the occasion of his 60th birthday

Abstract

Let G be a torsion-free hyperbolic group and let $n \geq 6$ be an integer. We prove that G is the fundamental group of a closed aspherical manifold if the boundary of G is homeomorphic to an $(n-1)$-dimensional sphere.

Introduction

If G is the fundamental group of an n-dimensional closed Riemannian manifold with negative sectional curvature, then G is a hyperbolic group in the sense of Gromov (see for instance [**6**, **7**, **21**, **22**]). Moreover, such a group is torsion-free and its boundary ∂G is homeomorphic to a sphere. This leads to the natural question whether a torsion-free hyperbolic group with a sphere as boundary occurs as a fundamental group of a closed aspherical manifold (see Gromov [**23**, page 192]). We settle this question if the dimension of the sphere is at least 5.

Theorem A. *Let G be a torsion-free hyperbolic group and let n be an integer ≥ 6. The following statements are equivalent:*

(i) The boundary ∂G is homeomorphic to S^{n-1}.
(ii) There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.

The aspherical manifold M appearing in our result is unique up to homeomorphism. This is a consequence of the validity of the Borel Conjecture for hyperbolic groups [**2**]; see also Section 3.

The proof depends on the surgery theory for homology ANR-manifolds due to Bryant, Ferry, Mio, and Weinberger [**9**] and the validity of the K- and L-theoretic Farrell-Jones Conjecture for hyperbolic groups due to Bartels, Reich, and Lück [**4**] and Bartels-Lück [**2**]. It seems likely that this result holds also if $n = 5$. Our methods can be extended to this case if the surgery theory from [**9**] can be extended to the case of

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