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A RIEMANNIAN BIEBERBACH ESTIMATE

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Abstract

The Bieberbach estimate, a pivotal result in the classical theory of univalent functions, states that any injective holomorphic function f on the open unit disc D satisfies $|f''(0)| \leq 4|f'(0)|$. We generalize the Bieberbach estimate by proving a version of the inequality that applies to all injective smooth conformal immersions $f : D \to \mathbb{R}^n$, $n \geq 2$. The new estimate involves two correction terms. The first one is geometric, coming from the second fundamental form of the image surface $f(D)$. The second term is of a dynamical nature, and involves certain Riemannian quantities associated to conformal attractors. Our results are partly motivated by a conjecture in the theory of embedded minimal surfaces.

1. Introduction

A conformal orientation-preserving local diffeomorphism that is defined in the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and takes values into $\mathbb{R}^2$ can be viewed as a holomorphic function $f : D \to \mathbb{C}$. Of special interest is the case when f is univalent, that is, injective. The class S of all holomorphic univalent functions in D satisfying $f(0) = 0$ and $f'(0) = 1$ was the object of much study in the last century, culminating with the solution by de Branges [**4**, **22**] of the celebrated Bieberbach conjecture: for any $f \in S$, the estimate

$$|f^{(k)}(0)| \leq kk! \tag{1.1}$$

holds for all $k \geq 2$. Equivalently, $|f^{(k)}(0)| \leq kk!|f'(0)|$ for any injective holomorphic function on D. The case $k = 2$, due to Bieberbach, yields the so-called distortion theorems which, in turn, imply the compactness of the class S [**19**, **22**]. Thus, the basic estimate $|f''(0)| \leq 4$ for $f \in S$, most commonly written in the form $|a_2| \leq 2$ where $f(z) = z + a_2 z^2 + \cdots$, already yields important qualitative information. In particular, it follows from the compactness of S that there are constants C_k such that $|f^{(k)}(0)| \leq C_k|f'(0)|$ for every $k \geq 2$ and injective holomorphic function f on D. The Bieberbach conjecture (the de Branges theorem) asserts that one can take C_k to be $kk!$.

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