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CALABI FLOW AND PROJECTIVE EMBEDDINGS

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Abstract

Let $X \subset \mathbb{CP}^N$ be a smooth subvariety. We study a flow, called balancing flow, on the space of projectively equivalent embeddings of X which attempts to deform the given embedding into a balanced one. If $L \to X$ is an ample line bundle, considering embeddings via $H^0(L^k)$ gives a sequence of balancing flows. We prove that, provided these flows are started at appropriate points, they converge to Calabi flow for as long as it exists. This result is the parabolic analogue of Donaldson's theorem relating balanced embeddings to metrics with constant scalar curvature [**12**]. In our proof we combine Donaldson's techniques with an asymptotic result of Liu and Ma [**17**] which, as we explain, describes the asymptotic behavior of the derivative of the map FS ∘ Hilb whose fixed points are balanced metrics.

1. Introduction

1.1. Overview of results. The idea of approximating Kähler metrics by projective embeddings goes back several years. The fundamental fact is that the projective metrics are dense in the space of all Kähler metrics. More precisely, let $L \to X^n$ be an ample line bundle over a complex manifold and let h be a Hermitian metric in L whose curvature defines a Kähler metric $\omega \in c_1(L)$. Together, h and ω determine an L^2-inner-product on the vector spaces $H^0(L^k)$. Using an L^2-orthonormal basis of sections for each $H^0(L^k)$ gives a sequence of embeddings $\iota_k \colon X \to \mathbb{CP}^{N_k}$ into larger and larger projective spaces and hence a sequence of projective metrics $\omega_k = \frac{1}{k}\iota_k^*\omega_{\mathrm{FS}}$ in the same cohomology class as ω.

Theorem 1 (Tian [**30**], Ruan [**28**]). *The metrics ω_k converge to ω in C^∞ as $k \to \infty$.*

(The sequence ω_k was considered by Yau in the case when ω is Kähler–Einstein [**35**]. Tian proved Theorem 1 with C^2-convergence; this was then improved to C^∞-convergence by Ruan.)

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