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CLASSIFICATION OF COMPACT ANCIENT SOLUTIONS TO THE CURVE SHORTENING FLOW

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Abstract

We consider an embedded convex compact ancient solution Γ_t to the curve shortening flow in $\mathbb{R}^2$. We prove that Γ_t is either a family of contracting circles, which is a type I ancient solution, or a family of evolving Angenent ovals, which is of type II.

1. Introduction

We consider an ancient embedded solution $\Gamma_t \subset \mathbb{R}^2$ of the curve shortening flow

$$\frac{\partial \mathbf{X}}{\partial t} = -\kappa\, \mathbf{N} \tag{1.1}$$

which moves each point $\mathbf{X}$ on the curve Γ_t in the direction of the inner normal vector $\mathbf{N}$ to the curve at P by a speed which is equal to the curvature κ of the curve.

In [**4**] Gage and Hamilton proved that if Γ_0 is a convex curve embedded in $\mathbb{R}^2$, then equation (1.1) shrinks Γ_t to a point. In addition, the curve remains convex and becomes asymptotically circular close to its extinction time.

In [**5**] Grayson studied the evolution of non-convex embedded curves under (1.1). He proved that if Γ_0 is any embedded curve in $\mathbb{R}^2$, the solution Γ_t does not develop any singularities before it becomes strictly convex.

Let $\Gamma_t \subset \mathbb{R}^2$ be an embedded ancient solution to the curve shortening flow (1.1). If s is the arclength along the curve and $\mathbf{X} = (x, y)$, we can express (1.1) as a system

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2}, \qquad \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial s^2}.$$

The evolution for the curvature κ of Γ_t is given by

$$\kappa_t = \kappa_{ss} + \kappa^3, \tag{1.2}$$

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