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A CHARACTERIZATION OF THE STANDARD EMBEDDINGS OF $\mathbb{C}P^2$ AND Q^3

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Abstract

H. Hopf showed that the only constant mean curvature sphere $\mathbb{S}^2$ immersed in $\mathbb{R}^3$ is the round sphere. The Kähler framework is an adequate approach to generalize Hopf' s theorem to higher dimensions. When $\varphi : M \to \mathbb{R}^n$ is an isometric immersion from a Kähler manifold, the complexified second fundamental form α splits according to types. The $(1,1)$ part of the second fundamental form plays the role of the mean curvature for surfaces and will be called the pluri-mean curvature *pmc*. Therefore isometric immersions with parallel pluri-mean curvature (*ppmc* isometric immersions) generalize in a natural way the cmc immersions. It is a standard fact that $\mathbb{R}^8$ is the smallest space where $\mathbb{C}P^2$ can be embedded. The aim of this work is to generalize Hopf's theorem proving in particular that the only *ppmc* isometric immersion from $\mathbb{C}P^2$ into $\mathbb{R}^8$ is the standard immersion.

1. Introduction and statement of results

The smallest $\mathbb{R}^k$ into which $\mathbb{S}^2 = \mathbb{C}P^1$ may be embedded is $\mathbb{R}^3$. H. Hopf [**13**] showed that, up to congruence, the only constant mean curvature (*cmc*) isometric immersion from the sphere into $\mathbb{R}^3$ is the standard immersion. Affording higher dimensions in the domain manifold, an adequate setting is the class of Kähler manifolds. When M is a Kähler manifold and $\varphi : M \longrightarrow \mathbb{R}^n$ is an isometric immersion, the coupling of the second fundamental form α of φ with the complex structure J of M originates two operators. To describe these operators we denote respectively by T^cM, $T'M$ and $T''M$ the complexification of TM and the eigenbundles of J corresponding to the eigenvalues i and $-i$. We will denote π' and π'' respectively the orthogonal projections of T^cM onto $T'M$ and $T''M$. Accordingly, each $X \in T^cM$ is decomposed as $X = X' + X''$ where

$$X' = \pi'(X) = \tfrac{1}{2}(X - iJX), \quad X'' = \pi''(X) = \tfrac{1}{2}(X + iJX)$$

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