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ON THE σ_2-SCALAR CURVATURE

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Abstract

In this paper, we establish an analytic foundation for a fully non-linear equation $\frac{\sigma_2}{\sigma_1} = f$ on manifolds with metrics of positive scalar curvature and apply it to give a (rough) classification of such manifolds. A crucial point is a simple observation that this equation is a degenerate elliptic equation without any condition on the sign of f and it is elliptic not only for $f > 0$ but also for $f < 0$. By defining a Yamabe constant $Y_{2,1}$ with respect to this equation, we show that a manifold with metrics of positive scalar curvature admits a conformal metric of positive scalar curvature and positive σ_2-scalar curvature if and only if $Y_{2,1} > 0$. We give a complete solution for the corresponding Yamabe problem. Namely, let g_0 be a positive scalar curvature metric, then in its conformal class there is a conformal metric with

$$\sigma_2(g) = \kappa\sigma_1(g),$$

for some constant κ. Using these analytic results, we give a rough classification of the space of manifolds with metrics of positive scalar curvature.

1. Introduction

Let (M, g_0) be a compact Riemannian manifold of dimension n with metric g_0 and $[g_0]$ the conformal class of g_0. Let Ric_g and R_g denote the Ricci tensor and scalar curvature of a metric g respectively. The Schouten tensor of the metric g is defined by

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right).$$

The importance of the Schouten tensor in conformal geometry can be viewed in the following decomposition of the Riemann curvature tensor

$$Riem_g = W_g + S_g \owedge g,$$

where $\owedge$ is the Kulkani-Nomizu product and W_g is the Weyl tensor. Note that $g^{-1} \cdot W_g$ is invariant in a given conformal class. Therefore, in a conformal class the Schouten tensor is important.

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