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COMPARISON GEOMETRY FOR THE BAKRY-EMERY RICCI TENSOR

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To the memory of Detlef Gromoll

Abstract

For Riemannian manifolds with a measure $(M, g, e^{-f} d\mathrm{vol}_g)$ we prove mean curvature and volume comparison results when the ∞-Bakry-Emery Ricci tensor is bounded from below and f or $|\nabla f|$ is bounded, generalizing the classical ones (i.e. when f is constant). This leads to extensions of many theorems for Ricci curvature bounded below to the Bakry-Emery Ricci tensor. In particular, we give extensions of all of the major comparison theorems when f is bounded. Simple examples show the bound on f is necessary for these results.

1. Introduction

In this paper we study smooth metric measure spaces $(M^n, g, e^{-f} d\mathrm{vol}_g)$, where M is a complete n-dimensional Riemannian manifold with metric g, f is a smooth real valued function on M, and $d\mathrm{vol}_g$ is the Riemannian volume density on M. These objects have been used extensively in geometric analysis and Kähler geometry, they play an essential role in Perelman's work on the Ricci flow, and they arise as smooth collapsed measured Gromov-Hausdorff limits. f is also referred to as the dilaton field in the physics literature. Smooth metric measure spaces are also called manifolds with density. In this paper by the Bakry-Emery Ricci tensor we mean

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f.$$

This is also referred to as the ∞-Bakry-Emery Ricci Tensor. Bakry and Emery [**4**] extensively studied (and generalized) this tensor and its relationship to diffusion processes. The Bakry-Emery tensor also occurs naturally in many different subjects, see e.g. [**24**] and [**31**, 1.3]. The equation $\mathrm{Ric}_f = \lambda g$ for some constant λ is exactly the gradient Ricci soliton equation, which plays an important role in the theory of Ricci

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