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EXTENSION OF TWISTED HODGE METRICS FOR KÄHLER MORPHISMS

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Abstract

Let $f : X \longrightarrow Y$ be a holomorphic map of complex manifolds, which is proper, Kähler, and surjective with connected fibers, and which is smooth over $Y \setminus Z$ the complement of an analytic subset Z. Let E be a Nakano semi-positive vector bundle on X. In our previous paper, we discussed the Nakano semi-positivity of $R^q f_*(K_{X/Y} \otimes E)$ for $q \geq 0$ with respect to the so-called Hodge metric, when the map f is smooth. Here we discuss the extension of the induced metric on the tautological line bundle $\mathcal{O}(1)$ on the projective space bundle $\mathbb{P}(R^q f_*(K_{X/Y} \otimes E))$ "over $Y \setminus Z$" as a singular Hermitian metric with semi-positive curvature "over Y". As a particular consequence, if Y is projective, $R^q f_*(K_{X/Y} \otimes E)$ is weakly positive over $Y \setminus Z$ in the sense of Viehweg.

1. Introduction

The subject in this paper is the positivity of direct image sheaves of adjoint bundles $R^q f_*\left(K_{X/Y} \otimes E\right)$, for a Kähler morphism $f : X \longrightarrow Y$ endowed with a Nakano semi-positive holomorphic vector bundle (E, h) on X. In our previous paper [**28**], generalizing a result in [**2**] in case $q = 0$, we obtained the Nakano semi-positivity of $R^q f_*\left(K_{X/Y} \otimes E\right)$ with respect to a canonically attached metric, the so-called Hodge metric, under the assumption that f is smooth. However the smoothness assumption on f is rather restrictive, and it is desirable to remove it. This is the aim of this paper.

To state our result precisely, let us fix notations and recall basic facts. Let $f : X \longrightarrow Y$ be a holomorphic map of complex manifolds. A real d-closed $(1,1)$-form ω on X is said to be *a relative Kähler form* for f, if for every point $y \in Y$, there exists an open neighbourhood W of y and a smooth plurisubharmonic function ψ on W such that $\omega + f^*(\sqrt{-1}\partial\overline{\partial}\psi)$ is a Kähler form on $f^{-1}(W)$. A morphism f is said to be *Kähler*, if there exists a relative Kähler form for f ([**35**, 6.1]), and $f : X \longrightarrow Y$ is said to be a *Kähler fiber space*, if f is proper, Kähler, and surjective with connected fibers.

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