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COMPLETE CONSTANT MEAN CURVATURE SURFACES AND BERNSTEIN TYPE THEOREMS IN $\mathbb{M}^2 \times \mathbb{R}$

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Abstract

In this paper we study constant mean curvature surfaces Σ in a product space, $\mathbb{M}^2 \times \mathbb{R}$, where $\mathbb{M}^2$ is a complete Riemannian manifold. We assume the angle function $\nu = \langle N, \frac{\partial}{\partial t} \rangle$ does not change sign on Σ. We classify these surfaces according to the infimum $c(\Sigma)$ of the Gaussian curvature of the projection of Σ.

When $H \neq 0$ and $c(\Sigma) \geq 0$, then Σ is a cylinder over a complete curve with curvature $2H$. If $H = 0$ and $c(\Sigma) \geq 0$, then Σ must be a vertical plane or Σ is a slice $\mathbb{M}^2 \times \{t\}$, or $\mathbb{M}^2 \equiv \mathbb{R}^2$ with the flat metric and Σ is a tilted plane (after possibly passing to a covering space).

When $c(\Sigma) < 0$ and $H > \sqrt{-c(\Sigma)}/2$, then Σ is a vertical cylinder over a complete curve of $\mathbb{M}^2$ of constant geodesic curvature $2H$. This result is optimal.

We also prove a non-existence result concerning complete multi-graphs in $\mathbb{M}^2 \times \mathbb{R}$, when $c(\mathbb{M}^2) < 0$.

1. Introduction

The image of the Gauss map of a complete minimal surface in $\mathbb{R}^3$ may determine the surface. For example, an entire minimal graph is a plane (Bernsteins' Theorem [**4**]). More generally, if the Gaussian image misses more than four points, then it is a plane ([**14**]). The Gaussian image of Scherks' doubly periodic surface misses exactly four points.

The image of the Gauss map of a non-zero constant mean curvature surface in $\mathbb{R}^3$ does determine the surface under certain circumstances. Hoffman, Osserman and Schoen proved (see [**16**]): *Let $\Sigma \subset \mathbb{R}^3$ be a complete surface of constant mean curvature. If the image of the Gauss map lies in an open hemisphere, then Σ is a plane. If the image is contained in a closed hemisphere, then Σ is a plane or a right cylinder.* Unduloids show that this result is the best possible.

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