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k-NORMAL SURFACES

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Abstract

Following Matveev, a *k-normal surface* in a triangulated 3-manifold is a generalization of both normal and (octagonal) almost normal surfaces. Using spines, complexity, and Turaev-Viro invariants of 3-manifolds, we prove the following results:

- a minimal triangulation of a closed irreducible or a bounded hyperbolic 3-manifold contains no non-trivial k-normal sphere;
- every triangulation of a closed manifold with at least 2 tetrahedra contains some non-trivial normal surface;
- every manifold with boundary has only finitely many triangulations without non-trivial normal surfaces.

Here, triangulations of bounded manifolds are actually *ideal* triangulations. We also calculate the number of normal surfaces of nonnegative Euler characteristics which are contained in the conjecturally minimal triangulations of all lens spaces $L_{p,q}$.

Introduction

We study in this paper the existence and number of (generalizations of) normal surfaces in various triangulated 3-manifolds. Theorem 1 below concerns the existence of (a generalization of) normal spheres in minimal triangulations. Theorems 2 and 3 then show that, with finitely many exceptions, all triangulations of a given manifold contain non-trivial normal surfaces. Finally, Theorem 5 calculates the number of normal surfaces of genus 0 and 1 in a family of triangulated lens spaces.

Normal surfaces. A *normal surface* in a triangulated 3-manifold is a surface intersecting each tetrahedron in triangles and squares, as in Fig. 1-(1). Since every incompressible surface (in an irreducible manifold) can be isotoped into normal position, normal surfaces can be used to investigate the set of incompressible surfaces algorithmically.

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