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COUNTING NODAL LINES WHICH TOUCH THE BOUNDARY OF AN ANALYTIC DOMAIN

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Abstract

We consider the zeros on the boundary $\partial\Omega$ of a Neumann eigenfunction φ_{λ_j} of a real analytic plane domain Ω. We prove that the number of its boundary zeros is $O(\lambda_j)$ where $-\Delta\varphi_{\lambda_j} = \lambda_j^2\varphi_{\lambda_j}$. We also prove that the number of boundary critical points of either a Neumann or Dirichlet eigenfunction is $O(\lambda_j)$. It follows that the number of nodal lines of φ_{λ_j} (components of the nodal set) which touch the boundary is of order λ_j. This upper bound is of the same order of magnitude as the length of the total nodal line, but is the square root of the Courant bound on the number of nodal components in the interior. More generally, the results are proved for piecewise analytic domains.

1. Introduction

This article is concerned with the high energy asymptotics of nodal lines of Neumann (resp. Dirichlet) eigenfunctions φ_{λ_j} on piecewise real analytic plane domains $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta\varphi_{\lambda_j} = \lambda_j^2\varphi_{\lambda_j} \text{ in } \Omega, \\ \\ \partial_\nu\varphi_{\lambda_j} = 0 \ \text{ (resp. } \varphi_{\lambda_j} = 0) \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

Here, ∂_ν is the interior unit normal. We refer to λ_j as the 'frequency' and note that the Laplace eigenvalue is λ_j^2 (unlike [**DF**, **L**] and some other references). We denote by $\{\varphi_{\lambda_j}\}$ an orthonormal basis of eigenfunctions of the boundary value problem corresponding to the eigenvalues $\lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \cdots$ enumerated according to multiplicity. The nodal set

$$\mathcal{N}_{\varphi_{\lambda_j}} = \{x \in \Omega : \varphi_{\lambda_j}(x) = 0\}$$

is a curve (possibly with self-intersections at the *singular points*) which intersects the boundary in the set $\mathcal{N}_{\varphi_{\lambda_j}} \cap \partial\Omega$ of boundary nodal points. The motivating problem of this article is the following: how many nodal

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