J. DIFFERENTIAL GEOMETRY
81 (2009) 631-647

THE KÄHLER-RICCI FLOW AND THE $\bar\partial$ OPERATOR ON VECTOR FIELDS

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Abstract

The limiting behavior of the normalized Kähler-Ricci flow for manifolds with positive first Chern class is examined under certain stability conditions. First, it is shown that if the Mabuchi K-energy is bounded from below, then the scalar curvature converges uniformly to a constant. Second, it is shown that if the Mabuchi K-energy is bounded from below and if the lowest positive eigenvalue of the $\bar\partial^\dagger\bar\partial$ operator on smooth vector fields is bounded away from 0 along the flow, then the metrics converge exponentially fast in C^∞ to a Kähler-Einstein metric.

1. Introduction

The Kähler-Ricci flow is arguably the most natural non-linear heat flow on a Kähler manifold, and its singularities and asymptotic behavior can be expected to provide a particularly deep probe of the geometry of the underlying manifold. For manifolds X with positive first Chern class, the Kähler-Ricci flow exists for all times [**C**], and the issue is its asymptotic behavior. The convergence of the flow would produce a Kähler-Einstein metric, the existence of which had been conjectured by Yau [**Y2**] to be equivalent to the stability of X in the sense of geometric invariant theory. Thus the convergence and, more generally, the asymptotic behavior of the flow should be related to stability conditions.

There have been, however, only relatively few results in this direction. In fact, the convergence of the flow for $c_1(X) > 0$ has been established only for $X = \mathbf{CP}^1$ [**H**, **Ch**, **CLT**], for X admitting a metric with positive bisectional curvature (and hence must be $\mathbf{CP}^n$) [**CT**], under the assumption that X already admits a Kähler-Einstein metric [**P2**] or a Kähler-Ricci soliton [**TZ2**], and for X toric with vanishing Futaki invariant [**Z**] (which is known to imply that X admits a Kähler-Einstein metric [**WZ**]). In [**PS**], it was shown that certain stability conditions do imply the convergence of the flow, without an a priori assumption

Research supported in part by National Science Foundation grants DMS-02-45371, DMS-06-04805, DMS-05-14003, and DMS-05-04285. Part of this work was carried out while the second-named author was supported by MSRI as a postdoctoral fellow.
Received 08/06/2007.