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LOWER SEMICONTINUITY OF THE WILLMORE FUNCTIONAL FOR CURRENTS

Reiner Schätzle

Abstract

The weak mean curvature is lower semicontinuous under weak convergence of varifolds, that is, if $\mu_k \to \mu$ weakly as varifolds then $\parallel \vec{\mathbf{H}}_\mu \parallel_{L^p(\mu)} \leq \liminf_{k\to\infty} \parallel \vec{\mathbf{H}}_{\mu_k} \parallel_{L^p(\mu_k)}$. In contrast, if $T_k \to T$ weakly as integral currents, then μ_T may not have a locally bounded first variation even if $\parallel \vec{\mathbf{H}}_{\mu_{T_k}} \parallel_{L^\infty(\mu_k)}$ is bounded.

In 1999, Luigi Ambrosio asked the question whether lower semicontinuity of the weak mean curvature is true when T is assumed to be smooth. This was proved in [**AmMa03**] for $p > n = \dim T$ in $\mathbb{R}^{n+1}$ using results from [**Sch04**]. Here we prove this in any dimension and codimension down to the desired exponent $p = 2$. For $p = n = 2$, this corresponds to the Willmore functional.

In a forthcoming joint work [**RoSch06**], main steps of the present article are used to prove a modified conjecture of De Giorgi that the sum of the area and the Willmore functional is the Γ-limit of a diffuse Landau-Ginzburg approximation.

1. Introduction

The Willmore functional of a surface immersed into Euclidian space is up to a factor the integral of the square mean curvature. For an integral $2-$varifold μ in $\mathbb{R}^m$ this extends to

$$\mathcal{W}(\mu) := \frac{1}{4} \int |\vec{\mathbf{H}}_\mu|^2 \, \mathrm{d}\mu.$$

We recall that the mean curvature of a submanifold is given in classical differential geometry as the trace of second derivatives. Elementary calculations show that the mean curvature determines the change of the area of the submanifold under local variations. In presence of singularities, this variational property is used to define the weak mean curvature,

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