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INSTABILITY OF GRAPHICAL STRIPS AND A POSITIVE ANSWER TO THE BERNSTEIN PROBLEM IN THE HEISENBERG GROUP $\mathbb{H}^1$

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Abstract

In the first Heisenberg group $\mathbb{H}^1$ with its sub-Riemannian structure generated by the horizontal subbundle, we single out a class of C^2 non-characteristic entire intrinsic graphs which we call strict graphical strips. We prove that such strict graphical strips have vanishing horizontal mean curvature (i.e., they are H-minimal) and are unstable (i.e., there exist compactly supported deformations for which the second variation of the horizontal perimeter is strictly negative). We then show that, modulo left-translations and rotations about the center of the group, every C^2 entire H-minimal graph with empty characteristic locus and which is not a vertical plane contains a strict graphical strip. Combining these results we prove the conjecture that in $\mathbb{H}^1$ the only stable C^2 H-minimal entire graphs, with empty characteristic locus, are the vertical planes.

1. Introduction

One of the most celebrated problems in geometry and calculus of variations is the Bernstein problem, which asserts that a C^2 minimal graph in $\mathbb{R}^3$ must necessarily be an affine plane. Following an old tradition, here minimal means of vanishing mean curvature. Bernstein [**Ber**] established this property in 1915. Almost fifty years later a new insight of Fleming [**Fle**] sparked a major development in the geometric measure theory which, through the celebrated works [**DG3**], [**Al**], [**Sim**], [**BDG**] culminated in the following solution of the Bernstein problem.

Theorem 1.1. *Let $\mathcal{S} = \{(x, u(x)) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n, x_{n+1} = u(x)\}$ be a C^2 minimal graph in $\mathbb{R}^{n+1}$, i.e., let $u \in C^2(\mathbb{R}^n)$ be a solution of the*

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