J. DIFFERENTIAL GEOMETRY
2 (1968) 77-114

HOMOGENEOUS SPACES DEFINED BY LIE GROUP AUTOMORPHISMS. I

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1. Introduction

This paper is a study of the structure and geometry of coset spaces $X = G/K$ where G is a reductive Lie group and K is an open subgroup of the fixed point set G^θ of a semisimple automorphism θ of G. The symmetric spaces are the case $\theta^2 = 1$. There the structure and classification theory for θ is well known, and the geometry of X basically comes down to a knowledge of the linear isotropy representation of K and the "Cartan decomposition" of the Lie algebra of G into eigenspaces of θ. We follow this general outline, starting with a structure theory for θ, obtaining full classifications (including the linear isotropy representations) in the cases which we know to have significant geometric interest, and then turning to geometric applications utilizing the θ-eigenspace decomposition of the Lie algebra. The geometric applications which we pursue are concerned with G-invariant almost complex structures and almost hermitian metrics on X. The almost complex structures themselves are used as a technical tool in passing from compact G to reductive G in the structure theory for θ.

§2 gives the structure theory for an inner automorphism θ of a compact Lie algebra $\mathfrak{G}$. Choosing a Cartan subalgebra and simple root system for $\mathfrak{G}$, we obtain a normal form for θ (Proposition 2.6). This gives us a simple root system for the fixed point set $\mathfrak{G}^\theta$ (Proposition 2.8), a criterion for whether $\mathfrak{G}^\theta$ is the centralizer of a torus (Proposition 2.11), and a method for enumerating all θ of any given fixed finite order (Proposition 2.11). §3 applies these results to a classification of inner automorphisms of order 3 (Theorem 3.3) and a classification of subalgebras $\mathfrak{G}^\theta \subset \mathfrak{G}$ which are not centralizers of tori (Theorem 3.5).

§4 is a complete classification and structure theory for invariant almost complex structures on coset spaces G/K where K is a connected[1] subgroup of

Received August 29, 1967. The work of the first author was supported by an Alfred P. Sloan Research Fellowship and National Science Foundation Grant GP-5798, and that of the second author by a National Science Foundation Postdoctoral Fellowship and National Science Foundation Grant GP-3990.

Due to its length this paper is being published in two parts; the second part will appear in the next issue.

[1] We show that the existence of an invariant almost complex structure on G/K implies connectedness of K.