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A NOTE ON CURVATURE AND FUNDAMENTAL GROUP

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Define the *growth function* γ associated with a finitely generated group and a specified choice of generators $\{g_1, \cdots, g_p\}$ for the group as follows (compare [9]). For each positive integer s let $\gamma(s)$ be the number of distinct group elements which can be expressed as words of length $\leq s$ in the specified generators and their inverses. (For example, if the group is free abelian of rank 2 with specified generators x and y, then $\gamma(s) = 2s^2 + 2s + 1$.) We will see that the asympotic behavior of $\gamma(s)$ as $s \to \infty$ is, to a certain extent, independent of the particular choice of generators (Lemma 1).

This note will make use of inequalities relating curvature and volume, due to R. L. Bishop [1], [2] and P. Günther [3], to prove two theorems.

Theorem 1. *If M is a complete n-dimensional Riemannian manifold whose mean curvature tensor R_{ij} is everywhere positive semidefinite, then the growth function $\gamma(s)$ associated with any finitely generated subgroup of the fundamental group $\pi_1 M$ must satisfy*

$$\gamma(s) \leq \text{constant} \cdot s^n .$$

It is conjectured that the group $\pi_1 M$ itself must be finitely generated.

The constant in this inequality will depend, of course, on the particular set of generators which is used to define $\gamma(s)$.

Theorem 2. *If M is compact Riemannian with all sectional curvatures less than zero, then the growth function of the fundamental group $\pi_1 M$ is at least exponential:*

$$\gamma(s) \geq a^s$$

for some constant $a > 1$.

In both cases, any set of generators for the group may be used in defining $\gamma(s)$.

Remarks. Note that there is always an exponential upper bound for $\gamma(s)$. In fact the inequality

$$\gamma(s + t) \leq \gamma(s)\gamma(t)$$

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