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GEODESIC LENGTH FUNCTIONS AND TEICHMÜLLER SPACES

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Abstract

Given a compact orientable surface with finitely many punctures Σ, let $\mathcal{S}(\Sigma)$ be the set of isotopy classes of essential unoriented simple closed curves in Σ. We determine a complete set of relations for a function from $\mathcal{S}(\Sigma)$ to $\mathbf{R}$ to be the geodesic length function of a hyperbolic metric with geodesic boundary and cusp ends on Σ. As a consequence, the Teichmüller space of hyperbolic metrics with geodesic boundary and cusp ends on Σ is reconstructed explicitly from an intrinsic $(\mathbf{Q}P^1, PSL(2, \mathbf{Z}))$ structure on $\mathcal{S}(\Sigma)$.

0. Introduction

Let $\Sigma = \Sigma_{g,r}^s$ be a compact oriented surface of genus g with r boundary components and s punctures, i.e., a surface of signature (g, r, s) where $(g, r, s) \geq 0$. The Teichmüller space of isotopy classes of hyperbolic metrics with geodesic boundary and cusp ends on Σ is denoted by $T_{g,r}^s = T(\Sigma)$, and the isotopy classes of essential simple closed unoriented curves in Σ is denoted by $\mathcal{S} = \mathcal{S}(\Sigma)$. A simple loop in Σ is called *parabolic* if it is homotopic into an end of Σ. The set of isotopy classes of essential parabolic simple loops in Σ is denoted by $P(\Sigma)$. For each $m \in T(\Sigma)$ and $\alpha \in \mathcal{S}(\Sigma)$, let $l_m(\alpha)$ be the length of the geodesic representing α if $\alpha \notin P(\Sigma)$ and let $l_m(\alpha) = 0$ if $\alpha \in P(\Sigma)$. The goal of the paper is to characterize the geodesic length function l_m in terms of an intrinsic $(\mathbf{Q}P^1, PSL(2, \mathbf{Z}))$ structure on $\mathcal{S}(\Sigma)$.

Theorem 1. *For surface $\Sigma_{g,r}^s$ of negative Euler number, a function $f : \mathcal{S}(\Sigma_{g,r}^s) \to \mathbf{R}$ is a geodesic length function if and only if $f|_{\mathcal{S}(\Sigma')}$ is a geodesic length function for each incompressible subsurface $\Sigma' \cong \Sigma_{1,1}^0$, $\Sigma_{0,r}^s$ $(r + s = 4)$ in $\Sigma_{g,r}^s$. Furthermore, geodesic length functions on*

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