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ON THE STRUCTURE OF SPACES WITH RICCI CURVATURE BOUNDED BELOW. I

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0. Introduction

In this paper and in [12], [13], we study the structure of spaces, Y, which are pointed Gromov-Hausdorff limits of sequences, $\{(M_i^n, p_i)\}$, of complete, connected Riemannian manifolds whose Ricci curvatures have a definite lower bound, say $Ric_{M_i^n} \geq -(n-1)$. In Sections 5–7, and sometimes in [12], we also assume a lower volume bound, $\mathrm{Vol}\,(B_1(p_i)) \geq v > 0$. In this case, the sequence is said to be *noncollapsing*. If $\lim_{i\to\infty} \mathrm{Vol}\,(B_1(p_i)) = 0$, then the sequence is said to *collapse*. It turns out that a convergent sequence is noncollapsing if and only if the limit has positive n-dimensional Hausdorff measure. In particular, any convergent sequence is either collapsing or noncollapsing. Moreover, if the sequence is collapsing, it turns out that the Hausdorff dimension of the limit is actually $\leq n-1$; see Sections 3 and 5.

Our theorems on the infinitesimal structure of limit spaces have equivalent statements in terms of (or implications for) the structure on a small but definite scale, of manifolds with $Ric_{M^n} \geq -(n-1)$. Although both contexts are significant, for the most part, it is the limit spaces which are emphasized here. Typically, the relation between corresponding statements for manifolds and limit spaces follows directly from the continuity of the geometric quantities in question under Gromov-Hausdorff limits, together with Gromov's compactness theorem, [37]; Theorems 2.45, 5.12 (see also Remark 5.13), 7.5, 7.6, are examples of

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