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AN INDEX THEOREM FOR FAMILIES OF DIRAC OPERATORS ON ODD-DIMENSIONAL MANIFOLDS WITH BOUNDARY

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Abstract

For a family of Dirac operators, acting on Hermitian Clifford modules over the odd-dimensional compact manifolds with boundary which are the fibres of a fibration with compact base, we compute the Chern character of the index, in K^1 of the base. Although we assume a product decomposition near the boundary, we make no assumptions on invertibility of the boundary family and instead obtain a family of self-adjoint Fredholm operators by choice of an auxiliary family of projections respecting the $\mathbb{Z}_2$ decomposition of bundles over the boundary. In case the boundary family is invertible, this projection can be taken to be the Atiyah-Patodi-Singer projection and the resulting formula is as conjectured by Bismut and Cheeger. The derivation of the index formula is effected by the combination of the superconnection formalism of Quillen and Bismut, the calculus of b-pseudodifferential operators and suspension.

Introduction

Let $\phi : M \longrightarrow B$ be a fibration of Riemannian manifolds, with B compact and with fibres diffeomorphic to a fixed *odd-dimensional* compact manifold with boundary X. Suppose that the fibres carry smoothly varying spin structures and that the Riemannian metrics on the fibres have smoothly varying product decompositions near the boundary. Let $\eth = \eth_z$ be, for $z \in B$, the associated family of Dirac operators and let $\eth_0 = \eth_{0,z}$ be the boundary family. If $\eth_{0,z}$ is invertible for each $z \in B$, the Atiyah-Patodi-Singer boundary condition makes $\eth_z$ into a

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