

FAMILIES OF DIRAC OPERATORS, BOUNDARIES AND THE B -CALCULUS

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Abstract

A version of the Atiyah-Patodi-Singer index theorem is proved for general families of Dirac operators on compact manifolds with boundary. The vanishing of the analytic index of the boundary family, in K^1 of the base, allows us to define, through an explicit trivialization, a smooth family of boundary conditions of generalized Atiyah-Patodi-Singer type. The calculus of b -pseudodifferential operators is then employed to establish the family index formula. A relative index formula, describing the effect of changing the choice of the trivialization, is also given. In case the boundary family is invertible the form of the index theorem obtained by Bismut and Cheeger is recovered.

Introduction

Let $\phi : M \rightarrow B$ be a smooth fibration of a manifold with boundary, M , with compact fibres diffeomorphic to a fixed manifold with boundary, X . In case the fibres are even-dimensional, carry smoothly varying spin structures and metrics which are of product type near the boundary and the Dirac operators induced on the fibres of the boundary fibration are all invertible, Bismut and Cheeger [9] obtained a family version of the Atiyah-Patodi-Singer index theorem:

$$(1) \quad \text{Ch}(\text{Ind}) = \phi_*(\widehat{A}) - \frac{1}{2}\widehat{\eta}.$$

Here $\text{Ch}(\text{Ind})$ is the Chern character, in $H^{\text{ev}}(B)$, of the virtual bundle formed by the \mathbb{Z}_2 -graded null spaces of the family of Dirac operators,

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